

Hartogs extension for systems of differential equations

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Abstract. The phenomenon of removable singularity is studied for overdetermined elliptic systems of differential equations. We show that the dimension of the characteristic variety of the system plays a key role in the problem.

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1 Introduction

According to the famous theorems of Hartogs and Osgood-Brown any compact singularity (with no holes) of a holomorphic function of several variables is removable. This fact can be viewed as a property of solutions of the Cauchy-Riemann system of differential equations in a domain of the real space \mathbb{R}^{2m} where $m > 1$. This phenomenon was extended by Ehrenpreis [4] who stated that a compact singularity is always removable for any system of two equations with constant coefficients and relatively prime symbols. It was shown in [9] that the automatic extension property of solutions of a general system M of equations with constant coefficients is governed by the modules $\text{Ext}^k(M, D)$, $k = 1, 2, \dots$ (see below). In particular the equation $\text{Ext}^1 = 0$ guarantees absence of compact singularity. Vanishing of higher Ext implies removing of some non-compact singularities in particular for singularities supported by submanifolds. Kawai [13] and Kawai and Takei [15] treated systems of equations with one unknown function and commuting operators. They stated automatic extension of solutions in the class of hyperfunctions. According to [13] (Theorem 2) the last property is however weaker than automatic extensibility in the classical sense.

We study here the phenomenon of compulsory extension for systems of linear partial differential equations with analytic coefficients in \mathbb{R}^n of the general form

$$P(x, \partial_x)u = 0 \quad (1)$$

Here $\partial_x \doteq (\partial/\partial x_1, \dots, \partial/\partial x_n)$, $P = \{p_{ij}\}$ is a $s \times r$ -matrix differential operator, $u = (u_1, \dots, u_r)$ are unknown functions, numbers s and r are arbitrary. We show that for any elliptic system a small compact singularity is always removable (in the classical sense) if dimension d of the characteristic variety V is strictly less than $n - 1$ just as for the case of Cauchy-Riemann system with $m > 1$. In the case of smaller d a more strong statement holds (Theorem 17). In particular, an

analytic submanifold S of dimension s can not be a support of a nonremovable singularity of a solution if $s < n - d - 1$. This is not the case if $s = n - d - 1$.

2 Regularity conditions for a differential matrix

We do not assume any special structure of the matrix P but impose a general condition of regularity which is well known in several forms [1],[8]. Fix an arbitrary point $x \in X$, denote by O_x the algebra of germs of analytic functions at the point $x \in \mathbb{R}^n$ and by D_x the algebra of differential operators in O_x . Let $P_i = (p_{i1}, \dots, p_{ir})$, $i = 1, \dots, s$ be rows of the matrix P ; consider a linear combination

$$Q(x, \partial_x) = \sum_{i=1}^s a_i(x, \partial_x) P_i(x, \partial_x) \in D_x^r \quad (2)$$

with some $a_i \in D_x$ where $Q = (q_1, \dots, q_r)$. We assume that there exist $b_i \in D_x$, $i = 1, \dots, s$ such that

$$Q(x, \partial_x) = \sum b_i(x, \partial_x) P_i(x, \partial_x) \quad (3)$$

and $\text{ord } b_i + \text{ord } p_{ij} \leq \text{ord } q_j$ for all $i = 1, \dots, s$, $j = 1, \dots, r$ (orda means the order of a differential operator a). In other words, there is no cancellations of higher order terms in the right-hand side of (3). This condition is not in fact restrictive since it can be always satisfied if the matrix P is supplemented by several lines of the form (2).

Definition. Fix some integers $\sigma_1, \dots, \sigma_s$ and ρ_1, \dots, ρ_r (called shifts) such that

$$\deg p_{ij} \leq \sigma_i - \rho_j, \quad i = 1, \dots, s; \quad j = 1, \dots, r$$

The principal part of the system (à la Douglis-Nirenberg [3]) is the matrix $P = \{p_{ij}\}$, where p_{ij} is the sum of homogeneous terms of p_{ij} of degree $\sigma_i - \rho_j$ ($p_{ij} = 0$ if there is no such terms). Substituting partial derivatives $\partial/\partial x_i$ by independent variables ξ_i , $i = 1, \dots, n$, we obtain homogeneous polynomials $p_{ij}(x, \xi)$ in $\xi = (\xi_1, \dots, \xi_n)$ with coefficients in O_x . The next condition is essential:

(*) For any point $x \in X$ and polynomials $r_1, \dots, r_s \in \mathbb{C}[\xi_1, \dots, \xi_n]$ such that

$$\sum_i r_i(\xi) P_i(x, \xi) = 0$$

where $P_i(x, \xi)$ denotes the vector $(p_{i1}(x, \xi), \dots, p_{ir}(x, \xi))$ there exist functions $R_1, \dots, R_s \in O_x[\xi_1, \dots, \xi_n]$ such that

$$\sum_i R_i(y, \xi) P_i(y, \xi) = 0$$

for y in a neighborhood of the point x such that and $R_i(x, \xi) = r_i(\xi)$, $i = 1, \dots, s$. In fact, this condition need to be checked for only finite number of vectors (r_1, \dots, r_s) and it is generic that is (**) is always fulfilled in the complement to a nowhere dense analytic set [10].

Note that in the case $r = s = 1$ the condition (*) only means that the principal part P of P does not vanish at x .

3 Differential modules and filtrations

Now we rewrite the above conditions in a more algebraic form. Let again $x \in \mathbb{R}^n$ and D be the algebra of differential operators in \mathbb{R}^n with coefficients in the algebra O of germs at x of analytic functions in \mathbb{R}^n (here and later we omit the subscript x). The algebra D has natural filtration $\{D_k, k = 0, 1, \dots\}$, where D_k is the O -module of differential operators $a \in D$ of order $\text{ord } a \leq k$ and $D_0 = O$. The associated graded module

$$\text{gr } D = \bigoplus_{k=0}^{\infty} D_k / D_{k-1}$$

is a commutative O -algebra. Fix a coordinate system x_1, \dots, x_n in \mathbb{R}^n . The algebra D is isomorphic to the graded algebras $O[\xi_1, \dots, \xi_n]$ where the generator ξ_i is represented by the operator $\partial/\partial x_i, i = 1, \dots, n$. The algebra $D \otimes_O \mathbb{C}$ is then isomorphic to the graded algebra of homogeneous polynomials in $T_x^*(\mathbb{R}^n)$.

Fix a natural r and a vector $\rho = (\rho_1, \dots, \rho_r) \in \mathbb{Z}^r$; the increasing sequence of O -submodules

$$D_k^\rho = \{a \in D^r, \text{ord}_\rho a \leq k\}, k \in \mathbb{Z}$$

is called filtration generated by the shift vector ρ , where $\text{ord}_\rho a = \text{ord}_\rho(a_1, \dots, a_r) = \max_i \text{ord } a_i + \rho_i$. The graded vector space

$$D^\rho = \bigoplus_k D_k^\rho / D_{k-1}^\rho$$

is a module over the graded commutative algebra D . Let r, s be natural numbers; any morphism of left D -modules $P : D^s \rightarrow D^r$ can be written in the form $a \mapsto aP$, where an element $a = (a_1, \dots, a_s) \in D^s$ is thought as row and P as a $s \times r$ -matrix whose entries $p_{ij}, i = 1, \dots, s, j = 1, \dots, r$ are sections of D . Let σ denote the filtration in D^s defined by a shift vector $\sigma = (\sigma_1, \dots, \sigma_s)$. The morphism P agrees with the filtrations, if $\text{ord}_\rho(aP) \leq \text{ord}_\sigma a$ for any $a \in D^s$. This condition is equivalent to the inequalities $\text{ord } p_{ij} \leq \sigma_i - \rho_j$. Let p_{ij} be the some of terms of p_{ij} of order $\sigma_i - \rho_j$. The matrix $P = \{p_{ij}\}$ is called the principal part of P with respect to the filtrations generated by ρ and σ . The operator P is called *elliptic* at a point x if $\text{rank } P(x, \xi) = s$ for any real $\xi \neq 0$.

Let M be a left D -module; suppose that M has a increasing filtration by O -submodules $M_k, k \in \mathbb{Z}$ such that $\bigcup M_k = M$ and $D_i M_k \subset M_{k+i}$ for any i and k . Then we call M filtered D -module. For such a module the direct sum

$$\text{gr } M = \bigoplus_{k=-\infty}^{\infty} M_k / M_{k-1}$$

has a natural structure of D -module.

Definition. Let M and N be filtered left (or right) D -modules. We say that a D -morphism $\alpha : M \rightarrow N$ agrees with the filtrations, if $\alpha(M_k) \subset N_k$ for $k = 0, 1, 2, \dots$. If α agrees with filtrations it generates a morphism of graded modules $\text{gr } \alpha : \text{gr } M \rightarrow \text{gr } N$ and the correspondence $\alpha \mapsto \text{gr } \alpha$ is a functor.

4 Complexes and symbols

Let

$$\dots \rightarrow D^t \xrightarrow{Q} D^s \xrightarrow{P} D^r \quad (4)$$

be an exact sequence of left D -modules. The morphism P in (14) acts by right multiplication $a \mapsto aP$ where P is a $s \times r$ -matrix whose entrees p_{ij} belong to D . The morphisms \dots, Q can be realized in a similar way.

Definition. We say that the modules in (4) are supplied with filtrations generated by some shift vectors $\rho, \sigma, \tau, \dots$. We shall denote these modules by $\dots, D^\tau, D^\sigma, D^\rho$ to fix the respective filtrations generated by some shift vectors $\dots, \tau, \sigma, \rho$ where in particular $\rho = (\rho_1, \dots, \rho_r)$, $\text{ord}_\rho a = \max_j \text{ord } a_j + \rho_j$. We assume that the morphisms \dots, Q, P agree with these filtrations which means $\dots, \text{ord } q_{jk} \leq \tau_j - \sigma_k, \text{ord } p_{ij} \leq \sigma_i - \rho_j$ for entrees of these matrices. The sequence of graded D -modules is then well-defined

$$\dots \rightarrow D^\tau \xrightarrow{Q} D^\sigma \xrightarrow{P} D^\rho$$

By tensoring over the algebra O we get the complex

$$\dots \rightarrow D^\tau \otimes \mathbb{C} \xrightarrow{Q \otimes \mathbb{C}} D^\sigma \otimes \mathbb{C} \xrightarrow{P \otimes \mathbb{C}} D^\rho \otimes \mathbb{C}$$

of free graded modules over the commutative algebra $A = D \otimes \mathbb{C} \cong \mathbb{C}[\xi_1, \dots, \xi_n]$. The set of maximal ideals in the algebra A is isomorphic to \mathbb{C}^n . For a maximal ideal \mathfrak{m} in A we take tensor product with the quotient algebra. This yields a complex of \mathbb{C} -vector spaces

$$\dots \rightarrow (D^\tau \otimes \mathbb{C}) \otimes_A A/\mathfrak{m} \xrightarrow{Q_{\mathfrak{m}}} (D^\sigma \otimes \mathbb{C}) \otimes_A A/\mathfrak{m} \xrightarrow{P_{\mathfrak{m}}} (D^\rho \otimes \mathbb{C}) \otimes_A A/\mathfrak{m}$$

where $\dots, P_{\mathfrak{m}} = P \otimes \mathbb{C} \otimes A/\mathfrak{m}$. Because of $A/\mathfrak{m} \cong \mathbb{C}$ this complex can be written in a simple form

$$\dots \rightarrow \mathbb{C}^t \xrightarrow{Q(x, \xi)} \mathbb{C}^s \xrightarrow{P(x, \xi)} \mathbb{C}^r \quad (5)$$

where ξ is the point in \mathbb{C}^n corresponding to the ideal \mathfrak{m} . Here $\dots, Q(x, \xi), P(x, \xi)$ are matrices whose entrees are analytic functions of x and polynomial function of ξ .

Definition. We call (5) the *principal symbol* of (4). The complex (4) is called elliptic if the symbol is exact at any real point $\xi \neq 0$.

5 Local solvability

Let $x \in \mathbb{R}^n$, $D = D_x$ and

$$\dots \rightarrow D^t \xrightarrow{Q} D^s \xrightarrow{P} D^r \quad (6)$$

be an exact sequence of left D -modules. Denote by E the space of germs at x of C^∞ -functions defined in \mathbb{R}^n . This space has the natural structure of a left

D -module. Applying the functor $\text{Hom}_D(\cdot, E)$ to (6) we obtain a complex of vector spaces:

$$E^r \xrightarrow{P} E^s \xrightarrow{Q} E^t \rightarrow \dots \quad (7)$$

where the matrices P, Q, \dots act by left multiplication like in (1).

Theorem 1 *If (6) is exact and elliptic, then the sequence (7) is exact.*

The case of arbitrary operator P with constant coefficients is considered in [5],[9]. For the case of analytic coefficients this statement is essentially due to Malgrange [11] who proved it for Newlander-Nirenberg's operator by reduction to the case of germs of analytic functions (the method coming from the Hodge theory). A proof in the general case was done by Andreotti-Nacinovich [14] by the same method. Quite different method was used by Hörmander [7].

We state here a quantitative version of this Theorem. Let E be the sheaf of germs of C^∞ -functions in \mathbb{R}^n . The space $E(U) = \Gamma(U, E)$ for any open $U \subset \mathbb{R}^n$ has the natural Fréchet topology. Any differential $s \times r$ -matrix P as in (1) defines for any open $U \subset X$ a linear continuous operator $P : E^r(U) \rightarrow E^s(U)$. We denote by $E_P(U)$ its kernel.

Fix an Euclidean structure in \mathbb{R}^n ; for a point $x \in \mathbb{R}^n$ and a number $r > 0$ the notation $U_x(r)$ means r -neighborhood of x .

Theorem 2 *If (6) is elliptic, then*

A. *there exists a continuous function b_x in X such that for arbitrary point $x \in X$, arbitrary $0 < r \leq 1$ and arbitrary $g \in E_Q(U_x(r))$ there exists a $f \in E^r(U_x(b_x r))$ such that*

$$Pf = g \quad (8)$$

in $U_x(b_x r)$.

B. *There exists a linear continuous operator $s_{x,r} : E_Q(U_x(r)) \rightarrow E^r(U_x(b_x r))$ that provides a solution to (8).*

◀ **1.** We will construct a Laplace-like operator Ω for (6) and reduce the statement to the case when $\Omega g = 0$. Because of the morphisms agree with the filtrations the inequalities are fulfilled

$$\text{ord } q_{ij} \leq \tau_i - \sigma_j, \text{ ord } p_{ij} \leq \sigma_i - \rho_j$$

where $Q = \{q_{ij}\}, P = \{p_{ij}\}$ are the entrees of matrices Q and P . For any differential operator a in $X \subset \mathbb{R}^n$ with analytic coefficients the formal adjoint operator a^* acts on smooth densities and on distributions with compact support in X :

$$\int_X a^*(v) u = \int_X v a(u)$$

Identifying a function u with the density $u dx$ where dx is the Euclidean volume form in \mathbb{R}^n we make the adjoint operator a^* acting on functions. It also has analytic coefficients. The transformation $a \mapsto a^*$ is \mathbb{C} -linear and we have $(ab)^* =$

b^*a^* . The Laplace operator Δ in \mathbb{R}^n is self-adjoint; we denote $\square = -\Delta$. For a natural k and a vector $\omega = (\omega_1, \dots, \omega_k)$ with natural coordinates we denote by \square^ω the diagonal $k \times k$ -matrix $(\square^{\omega_1}, \dots, \square^{\omega_k})$. Set $t = \max(\tau_1, \dots, \tau_t)$ and denote $t + \rho = (t + \rho_1, \dots, t + \rho_r)$, $t - \tau = \dots$. The differential operator

$$\Omega = P\square^{t+\rho}P^* + \square^\sigma Q^*\square^{t-\tau}Q\square^\sigma$$

is well defined in the sheaf E^s .

Lemma 3 *Ω is an elliptic operator in the sense of Douglis-Nirenberg in X with the shift vectors equal to $2t + 2\sigma = (2t + 2\sigma_1, \dots, 2t + 2\sigma_s)$.*

We postpone a proof of Lemma. Because of Ω is elliptic, there exists a countable family of local fundamental solutions Φ defined in open sets U_Φ such that $X = \cup U_\Phi$. Take a fundamental solution Φ (Lemma 4), an arbitrary point $x \in U_\Phi$ and a number $r > 0$ such that $U_x(r) \subset U_\Phi$. Choose a smooth cut function e with support in $U_x(r)$ that is equal to 1 in $U_x(r/2)$. Suppose that a function $g \in E^s(U_x(r))$ fulfils $Qg = 0$ and set

$$g_e = eg, \quad h = Q^*\square^{t-\tau}Q\square^\sigma\Phi g_e, \quad f = \square^{t+\rho}P^*\Phi g_e$$

We have

$$Pf = P\square^{t+\rho}P^*\Phi g_e = \Omega\Phi g_e - \square^\sigma Q^*\square^{t-\tau}Q\square^\sigma\Phi g_e = g_e - \square^\sigma h,$$

Thus the function f is a solution of (8) modulo a function $\square^\sigma h$. On the other hand

$$\Omega h = \square^\sigma Q^*\square^{t-\tau}Q\square^\sigma Q^*\square^{t-\tau}Q\square^\sigma\Phi g_e = \square^\sigma Q^*\square^{t-\tau}Q\Omega\Phi g_e = \square^\sigma Q^*\square^{t-\tau}Qg = 0$$

in $U_x(r/2)$ since $P^*Q^* = 0$ and $QP = 0$. It follows that the function h is analytic in $U_x(r/2)$ since of Petrowsky's result [2].

2. We show that h has holomorphic continuation in a quantified neighborhood Z_x of the point x .

Lemma 4 *Let A be a $s \times s$ -matrix differential operator with analytic coefficients in a ball $U \subset \mathbb{R}^n$ that is elliptic in the sense of Douglis-Nirenberg. Then there exists a fundamental solution $E = E(x, u)$ defined in $U \times U$ that admits a holomorphic extension $E(z, w)$ in the domain*

$$Z = \{z = x + iy, w = u + iv, c|y - v| \leq |x - u| < r\} \quad (9)$$

for some positive c and r . Moreover ϕE defines a bounded operator in $L_2(U) \rightarrow L_2(\mathbb{R}^n)$ for any test function ϕ with support in U .

◀ According to Douglis-Nirenberg's condition the operator A defines a map $A : D^\sigma \rightarrow D^\rho$ of order 0 where D^σ and D^ρ are free D -module of rank s with filtrations defined by some shift vectors ρ and σ . Let $A(x, D)$ be the principal part of this operator that is $A = \{a_{ij}\}$ where a_{ij} is the homogeneous part of

a_{ij} of order $\sigma_i - \rho_j$. The principal symbol $A(x, \xi)$ with respect to the vectors ρ, σ is elliptic for any $x \in X$ that is the scalar operator $\det A(x, \xi)$ is elliptic. Set $A_q(\partial_x) = A(q, \partial_x)$ and $B = A_q - A$ where q denotes the center of U . The elliptic operator A_q with constant coefficients possesses a fundamental solution $E_q(x, y) = E_q(x - y)$ in \mathbb{R}^n that has a holomorphic extension $E_0(z)$ to the neighborhood of $\mathbb{R}^n \setminus \{0\}$ of the form $\{z \in \mathbb{C}^n; |\operatorname{Im} z| < c_A |\operatorname{Re} z|\}$ where the constant c_A is determined from the condition $\det A_q(\xi + i\eta) \neq 0$ for $|\eta| < c_A |\xi|$ where $A_q(\zeta)$ is the symbol of A_q . Such a fundamental solution can be written as the Fourier-Laplace integral of $A_q^{-1}(\zeta)$ taken over a n -cycle in \mathbb{C}^n that coincides with \mathbb{R}^n up to a compact subset. We construct a fundamental solution for A by the method of E. Levi. Choose a number r that is smaller than the radius of U and take a test function ϕ in U that is equal to 1 for $|x - q| \leq r$. Consider the series of operators in $L_2(\mathbb{R}^n)$

$$E = E_q \sum_{k=0}^{\infty} F^k, \quad F = \phi B E_q \quad (10)$$

where we set $\phi B = 0$ in $\mathbb{R}^n \setminus U$. Let e_{ij} be the $s \times s$ -matrix whose entry equals 1 on ij -place and 0 otherwise and b be a differential operator in \mathbb{R}^n with constant coefficients such that $\operatorname{ord} b \leq \sigma_i - \rho_j$. The operator $b e_{ij}$ defines a map $D^\sigma \rightarrow D^\rho$ of order 0 which implies that the composition $b e_{ij} E_q$ is a bounded operator in $L_2(\mathbb{R}^n)^s$. We have

$$B = \sum_{i,j=1}^r \beta_{ij}(x) b_{ij} e_{ij}$$

where for all i, j , $\operatorname{ord} b_{ij} \leq \sigma_i - \rho_j$ and $\beta_{ij}(z)$ are analytic functions that vanish for $x = q$, since $B(q, \partial_x) = 0$. The norm $N = \|\phi B E_q\|$ can be made smaller 1/2 if we take $r = r_q$ sufficiently small and the series (10) converges as an operator in $L_2(U)^s$. Moreover we have

$$\|\psi E\| < 2 \|\psi E_q\| < \infty$$

for any test function ψ since the kernel E_q has weak singularity. It is easy to check that $AE = \operatorname{id}$ in U . The kernel $E(x, u)$ is real analytic out of the diagonal since the operator A is elliptic. Moreover it has a holomorphic extension to the domain (9) with $c = c_{A_q}/2$ due to Hörmander [6], Theorem 5.3.3. ►

Lemma 5 *For an arbitrary linear differential operator A with analytic coefficients in X that is elliptic in the sense of Douglis-Nirenberg there exists a positive continuous function r_q in X such that for any $q \in X$, any $r \leq r_q$ and arbitrary solution f of the equation $Af = 0$ in the ball $U_q(r)$ has a unique holomorphic extension to the ball $Z_q(c_q r)$ in \mathbb{C}^n with the same center and radius $c_q r$ where $c_q = c_{A_q}/2$ is a continuous function in X .*

◄ To prove the statement of Lemma we choose a number $r' < r$ and a cut function e supported in $U_q(r)$ that is equal to 1 in $U_q(r')$ and evaluate the

function f in $U_q(r')$ by the integral

$$f(x) = \int_{\mathbb{R}^n} A^*(u, \partial_u) e(u) f(u) E(x, u) du$$

We can move now the point x to an arbitrary point $z = x + \nu y$ such that $|y| < c_q r'/2$. The function $E(z, u)$ is holomorphic since the support of the integrand is contained in $U(r) \setminus U(r')$ hence $|x - u| > r'$. This gives a holomorphic extension of f in the ball $Z_q(c_q r/2)$. ►

Thus the function h as above has holomorphic extension to the ball $Z_x(b_x r)$ for some continuous function b_x . This construction has the property II with any $r < \text{dist}(x, \partial U_\Phi)$ and a positive continuous function $b_x = b_x(U_\Phi) \leq 1$ defined in the domain U_Φ . We set $b_x(U_\Phi) = 0$. Take the maximum

$$b_x(X) = \max_{\Phi} \left\{ \frac{\delta(x)}{\delta(x) + 1} b_x(U_\Phi), x \in U_\Phi \right\}$$

where $\delta(x) \doteq \text{dist}(x, \partial U_\Phi)$ for $x \in U_\Phi$ and $\delta(x) = 0$ for $x \in X \setminus U_\Phi$, over the family of all fundamental solutions Φ . It is well defined for any $x \in X$ and is a continuous function. We have $b_x(X) \leq b_x(U_\Phi)$ for any Φ , hence the function $b_x(X)$ fulfils **A** and **B**.

3. Now it is sufficient to prove the statements of Theorem 2 for the sheaf H of germs of analytic functions in \mathbb{R}^n . A construction of a solution to (8) in the space of germs of analytic functions can be done by the method of [10] that guarantees the properties **A**, **B** in terms of balls $Z_x(r)$ in \mathbb{C}^n . ►

4. Proof of Lemma 3. We have

$$\begin{aligned} \text{ord}(P \square^{t+\rho} P^*)_{ij} &\leq \max_k (\text{ord } p_{ik} + \text{ord } \square^{t+\rho_k} + \text{ord } p_{jk}) \\ &\leq \max_k (\sigma_i - \rho_k + \sigma_j - \rho_k + 2\rho_k + 2t) = \sigma_i + \sigma_j + 2t \end{aligned}$$

The same inequality holds for the matrix $\square^\sigma Q^* \square^{t-\tau} Q \square^\sigma$ and for Ω . The principal symbol $\Omega(z, \xi)$ of Ω with respect to the shift vector $2t + 2\sigma$ is equal to

$$\Omega = P R^{t+\rho} P^* + R^\sigma Q^* R^{t-\tau} Q R^\sigma$$

where R^ω means the symbol of the operator \square^ω . We will check that $\det \Omega(z, \xi) \neq 0$ as $\xi \in \mathbb{R}^n \setminus \{0\}$. If it is not the case for a point ξ then there exists a non zero vector $v \in \mathbb{R}^s$ such that $\Omega(z, \xi) v = 0$. Define the coordinate scalar product \langle, \rangle in \mathbb{R}^s and write

$$\begin{aligned} 0 &= \langle \Omega(x, \xi) v, v \rangle = \langle P R^{t+\rho} P^* v, v \rangle + \langle R^\sigma Q^* R^{t-\tau} Q R^\sigma v, v \rangle \\ &= \left\langle R^{(t+\rho)/2} P^* v, R^{(t+\rho)/2} P^* v \right\rangle + \left\langle R^{(t-\tau)/2+\sigma} Q v, R^{(t-\tau)/2+\sigma} Q v \right\rangle \end{aligned}$$

where $R^{\omega/2}$ mean a diagonal matrix with the positive diagonal terms $\sqrt{R^{\omega_i}}$, $i = 1, \dots, k$. Both terms in the right-hand side are non-negative, hence vanish. This yields

$$P^*(x, \xi) v = 0, \quad Q(x, \xi) v = 0 \quad (11)$$

By Proposition 12 the sequence of symbols (5) is exact at any real point (x, ξ) , $\xi \neq 0$. Therefore the first equation (11) implies that $v = Q^*(x, \xi) w$ for some vector $w \in \mathbb{R}^t$. By the second equation (11) we find $0 = \langle Q(x, \xi) v, w \rangle = \langle v, v \rangle$, that is $v = 0$. This contradicts to the assumption and completes the proof. ►

Corollary 6 *For any $x \in X$ and $r \leq 1$ there exist linear continuous operators $R_r : E^s(U_x(r)) \rightarrow E^r(U_x(b_x^3 r))$ and $\Sigma_r : E^t(U_x(r)) \rightarrow E^s(U_x(b_x^3 r))$ such that*

$$(PR_r + \Sigma_r Q)g = g \quad (12)$$

for any $g \in E^s(U_x(r))$.

◀ Write (6) with two more terms

$$\dots \rightarrow D^v \xrightarrow{S} D^u \xrightarrow{R} D^t \xrightarrow{Q} D^s \xrightarrow{P} D^r$$

and apply the functor $\text{Hom}_D(\cdot, E)$:

$$E^r \xrightarrow{P} E^s \xrightarrow{Q} E^t \xrightarrow{R} E^u \xrightarrow{S} E^v \rightarrow \dots$$

Here P, Q, R, S, \dots are linear operators as in (7). By Theorem 2 applied to these terms, there exist linear continuous operators

$$\begin{aligned} \rho_r : E_Q(U_x(r)) &\rightarrow E^r(U_x(b_x r)), \sigma_r : E_R(U_x(r)) \rightarrow E^s(U_x(b_x r)) \\ \tau_r : E_S(U_x(r)) &\rightarrow E^t(U_x(b_x r)) \end{aligned}$$

with the properties $P\rho_r f = f$, $Q\sigma_r g = g$, $R\tau_r h = h$ in $U_x(b_x r)$. We have $Q(g - \sigma_r Qg) = 0$ for any $g \in E^s(U_x(r))$. Therefore we can set $R_r g = \rho_{b^2 r}(g - \sigma_{br} Qg)$ and similarly $\Sigma_r h = \sigma_{br}(h - \tau_r R_r h)$. We have now for any $g \in E^s(U_x(r))$

$$\begin{aligned} (PR_r + \Sigma_r Q)g &= P\rho_{b^2 r}(g - \sigma_{br} Qg) + \sigma_{br}(Qg - \tau_r R_r Qg) \\ &= g - \sigma_{br} Qg + \sigma_{br} Qg = g \end{aligned}$$

and (12) follows. ►

6 Solutions with compact support

Let U be an open set in \mathbb{R}^n ; the topological dual space $E^*(U)$ to $E(U)$ is identified with space of distributions in \mathbb{R}^n with compact support contained in U . An arbitrary differential $s \times r$ -matrix P in U with analytic coefficients defines a continuous operator $P : E^r(U) \rightarrow E^s(U)$ and the adjoint operator $P^* : E^*(U)^s \rightarrow E^*(U)^r$ which acts by

$$P^* \phi = \psi, \quad \psi(u) = \phi(Pu), \quad \phi \in E^*(U)^s, \quad u \in E^s(U)$$

For any complex (6) of left D_X -modules in an open set $X \subset \mathbb{R}^n$ and any open set $U \subset X$ the sequence

$$\dots \rightarrow E^*(U)^t \xrightarrow{Q^*} E^*(U)^s \xrightarrow{P^*} E^*(U)^r$$

is a complex of vector spaces.

Theorem 7 *If (6) is an elliptic complex in an open set $X \subset \mathbb{R}^n$, then there exists a continuous function c_x in X such that*

C. *for any point $x \in X$ and any r , $0 < r \leq 1$ the kernel of $P^* : E^*(U_x(c_x r))^s \rightarrow E^*(U_x(c_x r))^r$ is contained in the image of $Q^* : E^*(U_x(r))^t \rightarrow E^*(U_x(r))^s$,*
D. *a function $\alpha \in E^*(U_x(c_x r))^r$ is equal to $P^*\beta$ for some $\beta \in E^*(U_x(r))^s$ if and only if $\alpha(u) = 0$ for any $u \in E_P(U_x(c_x r))$.*

◀ We omit the bottom index x . Dualizing (12) we get

$$R_r^* P^* \alpha + Q^* \Sigma_r^* \alpha = \alpha$$

for an arbitrary $\alpha \in E^*(U(cr))^s$. If $\alpha P = 0$, this equation yields $\alpha = Q\beta$, where $\beta = \Sigma_r^* \alpha \in E^*(U(r))^r$. This proves statement **C**.

Check **D**. If $\alpha = P^*\beta$, then $u(\alpha) = Pu(\beta) = 0$. Vice versa, let $u(\alpha) = 0$ for any $u \in E_P(U(cr))$. The distribution $\beta = \alpha - P^* R_r^* \alpha$ fulfils $v(\beta) = u(\alpha)$ where $w = v - R_r P v$ for an arbitrary $v \in E^s(U(r))$. We have $w \in E^s(U(cr))$ and $Pw = (P - PR_{cr}P)v = 0$ since of (12). Therefore $w(\alpha) = 0$, hence $v(\beta) = 0$ which yields $\beta = 0$ and $\alpha = P^*\gamma$, $\gamma = R_r^* \alpha$. ▶

7 Resolutions

Now we take more invariant point of view on systems of equations like (1).

Definition ([8]). Let M be a filtered left D -module and

$$\text{gr } M = \oplus_{k \in \mathbb{Z}} M_k / M_{k-1}$$

the corresponding graded D -module. We assume that

- (i) the D -module $\text{gr } M$ is finitely generated,
- (ii) the O -module $\text{gr } M_x$ is free.

Definition. Let M be a left D -module satisfying (i). The product $\text{gr } M \otimes_O \mathbb{C}$ is a module of finite type over the polynomial algebra $A = D \otimes \mathbb{C} \cong \mathbb{C}[\xi_1, \dots, \xi_n]$. The *characteristic* set of M is by definition the support of $V = V(M)$ in the support of the A -module $\text{gr } M \otimes_O \mathbb{C}$. The set V is an algebraic cone in the set \mathbb{C}^n of maximal ideals of the algebra A . Any point $\xi \in V$ can be interpreted as a multiplicative functional $\mu : \text{gr } M \otimes_O \mathbb{C} \rightarrow \mathbb{C}$ such that $\mu(am) = a(\xi)\mu(m)$ for arbitrary $a \in A, m \in \text{gr } M \otimes_O \mathbb{C}$.

We call M *elliptic* if the characteristic variety $V(M)$ contains no real point $\xi \neq 0$.

Remark. It is easy to check that the condition (ii) for $M = \text{Cok } P : D^\sigma \rightarrow D^\rho$ is equivalent to (*) for P . The characteristic set V of this module coincide with set of points ξ such that $\text{rank } P(x, \xi) < s$.

Definition. Let $\alpha : E \rightarrow F$ be a morphism of filtered D -modules. It is called *strict*, if it agrees with the filtrations and $\alpha(E_k) = \alpha(E) \cap F_k$, $k \in \mathbb{Z}$.

Proposition 8 *Let*

$$E \xrightarrow{\alpha} F \xrightarrow{\beta} G \tag{13}$$

be a complex of morphisms of filtered vector spaces. If $\text{Ker gr } \beta = \text{Im gr } \alpha$, the complex (13) is exact and α is strict.

◀ Let $\beta(f) = 0$ for an element $f \in F$. We have $f \in F_k$ for some k and $\text{gr } \beta(f) = 0$. By the condition there is an element $e_0 \in E_k$ such that $\text{gr } \alpha(e_0) = \text{gr } f$, that is $f - \alpha(e_0) \in F_{k-1}$. The element $f' \doteq f - \alpha(e_0)$ is contained in $\text{Ker } \beta$, we repeat the above arguments with k replaced by $k-1$ and obtain an element $e_1 \in E_{k-1}$ such that $f' - \alpha(e_1) \in F_{k-2}$ and so on. Finally we get $f = \alpha(e)$ where $\alpha = \alpha_0 + \alpha_1 + \dots \in E_k$. ▶

Proposition 9 *If the complex (13) is exact, α and β are strict, then $\text{Ker gr } \beta = \text{Im gr } \alpha$.*

◀ We have $\text{gr } \beta \text{ gr } \alpha = 0$. Show that $\text{Ker gr } \beta \subset \text{Im gr } \alpha$. Take an element $f \in F_k \doteq F_k/F_{k-1}$ such that $\text{gr } \beta(f) = 0$. Let $f \in F_k$ be an element of the class f . We have $\beta(f) \in G_{k-1}$ and $\beta(f) \in \beta(F_{k-1})$ since β is strict, that is $\beta(f - f') = 0$ for an element $f' \in F_{k-1}$. We have $f - f' = \alpha(e)$, $e \in E$ since (13) is exact and $\alpha(e) = \alpha(e')$ for some $e' \in E_k$ since α is strict. This yields $f = \text{gr } \alpha(e)$ where e is the class of e' . ▶

Let M be a filtered left D -module and

$$\dots \rightarrow D^\tau \xrightarrow{Q} D^\sigma \xrightarrow{P} D^\rho \xrightarrow{\pi} M \rightarrow 0 \quad (14)$$

be a strict exact sequence of filtered left D -modules. The complex of D -modules

$$\dots \rightarrow D^\tau \xrightarrow{Q} D^\sigma \xrightarrow{P} D^\rho \xrightarrow{\pi} \text{gr } M \rightarrow 0 \quad (15)$$

is then well defined where all morphisms have degree 0. We call (15) the *principal part* of (14).

Proposition 10 *If a left D -module M fulfils (i), then for any point $x \in X$ there exist a neighborhood U of x and a resolution of the graded module $\text{gr } M$.*

◀ The product $\text{gr } M \otimes_O \mathbb{C}$ is a module of the polynomial algebra $A \doteq D \otimes \mathbb{C}$. Construct a strict resolution of this module of the form

$$\dots \rightarrow D^\tau \otimes \mathbb{C} \xrightarrow{Q \otimes \mathbb{C}} D^\sigma \otimes \mathbb{C} \xrightarrow{P \otimes \mathbb{C}} D^\rho \otimes \mathbb{C} \xrightarrow{\pi} \text{gr } M \otimes \mathbb{C} \rightarrow 0 \quad (16)$$

By (i) there exists a surjective morphism $\pi : D^{r_0} \rightarrow \text{gr } M$. We choose a shift vector $\rho = (\rho_1, \dots, \rho_{r_0})$ where $\rho_i \doteq \text{ord } \pi_{\mathbb{C}}(e_i)$, $i = 1, \dots, r_0$ for the standard generators e_1, \dots, e_{r_0} and introduce the filtration D^ρ in the module D^{r_0} . The morphism $\pi : D^\rho \rightarrow \text{gr } M$ has degree 0 and generate A -morphism $\pi_A : A^\rho \rightarrow \text{gr } M \otimes \mathbb{C}$ where $A = D \otimes \mathbb{C}$. Because of the algebra A is Noetherian, the submodule $\text{Ker } \pi_A$ is generated by some homogeneous elements p_1, \dots, p_{r_1} . Let $P_A : A^{r_1} \rightarrow A^\rho$ be the morphism such that $P_A(e'_i) = p_i$, $i = 1, \dots, r_1$ for the standard generators e'_1, \dots, e'_{r_1} of A^{r_1} . Set $\sigma = (\sigma_1, \dots, \sigma_{r_1})$ where $\sigma_i \doteq \text{ord}_\rho p_i$ and introduce the filtration D^σ in D^{r_1} . The morphism P_A is homogeneous of degree 0 and $\text{Im } P_A = \text{Ker } \pi_A$. Because of the module D is Noetherian we can apply the same arguments to $\text{Ker } P_A$ and choose morphisms Q_A, \dots

By (ii) all the n morphisms P_A, Q_A, \dots have extensions to some D -morphisms P, Q, \dots such that the sequence (15) is a complex. It can be shown by standard homological arguments since the O -modules $\text{gr}M, D^\rho, D^\sigma, \dots$ are flat. ►

Proposition 11 *For any free graded resolution (15) there exists a free strict resolution (14) of M such that (15) is the principal part of (14).*

◀ Choose for any $i = 1, \dots, r_0$ an element $m_i \in M_{\rho_i}$ whose image in M_{ρ_i}/M_{ρ_i-1} is equal to $\pi(e_i)$ and define a D -morphism $\pi : D^{r_0} \rightarrow M$ such that $\pi(e_i) = m_i, i = 1, \dots, r_0$. This morphism agrees with the filtrations in D^ρ and in M and is surjective, since so is π . Next we lift P to a D -morphism $P_0 : D^{r_1} \rightarrow D^{r_0}$. For any standard generator e'_k of D^{r_1} the row $p_k \doteq P(e'_k) \in D^\rho$ satisfies $\pi p_k = 0$, which means $\pi p_k \in M_{\rho_k-1}, k = 1, \dots, r_1$. Because of exactness of (15), there exists an element $q_k \in D^{r_0}$ such that $\text{ord } q_k = \rho_k - 1$ and $\pi(q_k) = \pi p_k$. We have $\pi(p_k - q_k) \in M_{\rho_k-2}$ and so on up to filtration -1 . Finally we collect the lines $p_k - q_k - q'_k - \dots, k = 1, \dots, r_1$ in a matrix P of size $r_0 \times r_1$ and have $\pi P = 0$. The principal part of the line $P(e'_k)$ is equal to p_k , that is the principal part of P is P . By Proposition 8 P is strict and $\text{Im } P = \text{Ker } \pi$.

The image of the composition $PQ : D^{r_2} \rightarrow D^{r_0}$ is contained in $\text{Ker } \pi$ and $\text{ord}_\rho PQ(e'') < \text{ord}_\tau e''$ for each standard generator e'' of D^{r_2} . Because of $\text{Ker } \pi = \text{Im } P$, there exists an element $q_1 \in D^{r_1}$ such that $\text{ord}_\sigma q_1 = \text{ord}_\rho PQ(e'')$ and $PQ(e'') = Pq_1$ up to a term of filtration $< \text{ord}_\sigma q_1$. We make a matrix $Q_1 : D^{r_2} \rightarrow D^{r_1}$ from the lines $Q_1(e'') = q_1$ where e'' runs over the set of generators of D^{r_2} . Consider the composition $P(Q - Q_1) : D^{r_2} \rightarrow D^{r_0}$. We have now $\text{ord}_\rho P(Q - Q_1)(e'') < \text{ord}_\rho PQ(e'')$ and can find an element $q_2 \in D^{r_1}$ such that $\text{ord}_\sigma q_2 = \text{ord}_\rho P(Q - Q_1)(e'')$ up to a term of filtration $< \text{ord}_\sigma q_2$. Define a matrix Q_2 by $Q_2(e'') = q_2$ for the set of standard generators e'' then consider the matrix $Q - Q_1 - Q_2$ and so on. This series is finite since $\dots < \text{ord}_\sigma q_2 < \text{ord}_\sigma q_1 < \text{ord}_\tau e''$. We set $Q = Q - Q_1 - Q_2 - \dots$. By Proposition 8 P_1 is strict and $\text{Im } Q = \text{Ker } P$. We construct a matrix R such that $\text{Im } R = \text{Ker } Q$ in the similar way and so on. ►

Proposition 12 *If M is an elliptic module then any strict resolution resolution (14) of M is elliptic.*

◀ From (15) we get the sequence

$$\dots \rightarrow D^\tau \otimes \mathbb{C} \xrightarrow{Q \otimes \mathbb{C}} D^\sigma \otimes \mathbb{C} \xrightarrow{P \otimes \mathbb{C}} D^\rho \otimes \mathbb{C} \rightarrow \text{gr}M \otimes \mathbb{C} \rightarrow 0 \quad (17)$$

It is exact since of (ii). Take any real point $\xi \neq 0$; let \mathfrak{m} be the corresponding maximal ideal in A . Note that all the terms $\dots, D^\rho \otimes \mathbb{C}$ are flat over A and

$$\text{Tor}^*(\text{gr}M \otimes \mathbb{C}, A/\mathfrak{m}) = 0$$

since $(\text{gr}M \otimes \mathbb{C}) \otimes_A A/\mathfrak{m} = 0$ because of ξ does not belong to the characteristic set of $\text{gr}M \otimes \mathbb{C}$. Therefore tensoring (17) by A -module A/\mathfrak{m} we get the exact sequence

$$\dots \rightarrow D^\tau \otimes \mathbb{C} \otimes_A A/\mathfrak{m} \xrightarrow{Q(x, \xi)} D^\sigma \otimes \mathbb{C} \otimes_A A/\mathfrak{m} \xrightarrow{P(x, \xi)} D^\rho \otimes \mathbb{C} \otimes_A A/\mathfrak{m} \rightarrow 0$$

which proves the Proposition. ►

8 Key Lemma

Let (M, ρ) be a filtered D -module. The set $\text{Hom}_D(M, D)$ of D -morphisms $h : M \rightarrow D$ has a natural structure of two-side D -module since D has such a structure. It possesses the dual filtration ρ^* such that $\text{ord}_{\rho^*}(h) = k$ if $h(M_i) \subset D_{i+k}$ for any i . In particular $\text{Hom}(D^\rho, D) \cong D^{-\rho}$, where $D^{-\rho}$ is a free D -module of the same rank as D^ρ with the shift vector $-\rho$. Any morphism of left D -modules $P : D^\sigma \rightarrow D^\rho$ generates the *dual* morphism

$$P' \doteq \text{Hom}(P, D) : D^{-\rho} \rightarrow D^{-\sigma}, \quad h \mapsto Ph$$

where we interpret an element $h \in D^{-\rho}$ as a column. The map P' is a morphism of *right* D -modules.

Fix a point $x \in \mathbb{R}^n$. Let

$$R : \dots \rightarrow D^{\rho_2} \xrightarrow{P_1} D^{\rho_1} \xrightarrow{P_0} D^{\rho_0} \rightarrow 0 \quad (18)$$

be a strict resolution of a left D -module M where $\rho_0, \rho_1, \rho_2, \dots$ are some shift vectors. The complex $\text{Hom}_D(R, D)$ looks as

$$0 \rightarrow D^{-\rho_0} \xrightarrow{P'_0} D^{-\rho_1} \xrightarrow{P'_1} D^{-\rho_2} \rightarrow \dots \rightarrow D^{-\rho_{k-1}} \xrightarrow{P'_{k-1}} D^{-\rho_k} \rightarrow \dots \quad (19)$$

where $D^{-\rho_i}$ is a free right D -module of the same rank r_i as D^{ρ_i} and all morphisms agree with the filtrations and P' means left multiplication of a column by a matrix P . It is a complex of right D -modules.

Lemma 13 *If a left D -module M satisfies (i, ii), then the sequence (19) is exact at the terms $D^{-\rho_k}$ with $k = 0, \dots, m-1$ where $m = n - \dim_{\mathbb{C}} V(M)$.*

◀ The principal part of (19) is the complex of modules over the graded commutative algebra $D = D_x$:

$$\Pi : 0 \rightarrow D^{-\rho_0} \xrightarrow{P'_0} D^{-\rho_1} \rightarrow \dots \rightarrow D^{-\rho_{k-1}} \xrightarrow{P'_{k-1}} D^{-\rho_k} \rightarrow \dots$$

which is equal to $\text{Hom}_D(R, D)$ where R is the principal part of (18). We want to show that this complex is exact in terms with $k = 0, \dots, m-1$. By Proposition 9 R is a resolution of $\text{gr } M$. By the condition (ii) the complex $R \otimes \mathbb{C}$ is a resolution of $\text{gr } M \otimes \mathbb{C}$ over the algebra $A \doteq D \otimes \mathbb{C}$ where $\otimes = \otimes_{O_x}$ which is isomorphic to the graded polynomial \mathbb{C} -algebra $\mathbb{C}[\xi_1, \dots, \xi_n]$. This yields

$$H^k(\text{Hom}_A(R \otimes \mathbb{C}, A)) \cong \text{Ext}^k(\text{gr } M \otimes \mathbb{C}, A) \quad (20)$$

The right-hand side of (20) vanishes for $k = 0, \dots, m-1$ in virtue of [[9], Corollary 1, §13] which means that the complex $\text{Hom}_A(R \otimes \mathbb{C}, A)$;

$$0 \rightarrow D_x^{-\rho_0} \otimes \mathbb{C} \xrightarrow{P'_0 \otimes \mathbb{C}} D_x^{-\rho_1} \otimes \mathbb{C} \rightarrow \dots \rightarrow D_x^{-\rho_{k-1}} \otimes \mathbb{C} \xrightarrow{P'_{k-1} \otimes \mathbb{C}} D_x^{-\rho_k} \otimes \mathbb{C} \rightarrow \dots \quad (21)$$

is acyclic in terms $D_x^{-\rho_k} \otimes \mathbb{C}$, $k = 0, \dots, m-1$. By definition for any shift vector ω we have $D^\omega = \oplus_i D_i^\omega$ where D_i^ω is the vector space of homogeneous elements of grading i and $D^\omega \otimes \mathbb{C} = \oplus_i D_i^\omega \otimes \mathbb{C}$ where $D_i^\omega \otimes \mathbb{C}$ is a finite dimensional space. The complex (21) is isomorphic to $\Pi \otimes \mathbb{C}$ hence all morphisms are homogeneous of degree zero. Therefore

$$\Pi = \oplus \Pi_i, \quad \Pi \otimes \mathbb{C} = \oplus \Pi_i \otimes \mathbb{C}$$

where Π_i is a complex of free D -modules of finite type and $\Pi_i \otimes \mathbb{C}$ is a finite dimensional \mathbb{C} -vector spaces for each $i = 0, 1, \dots$. We know that the complex $\Pi_i \otimes \mathbb{C}$ is acyclic in degrees $< m$. By Nakayama's lemma the complex Π_i is acyclic in the same degrees for each i which implies that the same true for the complex Π . By Proposition 8 that (19) is also acyclic in degrees $k = 0, \dots, m-1$ and the morphisms P'_0, \dots, P'_{k-1} are strict. ►

Corollary 14 *The complex (19) is elliptic in any degree $k < m$. The statement C of Theorem 7 holds for this complex and a continuous function d_x in X .*

◄ By Lemma 13 we need only to check that for any real point $\xi \in \text{Spect} A \cong \mathbb{C}^n, \xi \neq 0$ the complex

$$\Pi_x \otimes A/\mathfrak{m}$$

is acyclic where $\mathfrak{m} \subset A$ is the maximal ideal of the point ξ . We have $H^*(\Pi_x \otimes A/\mathfrak{m}) \cong H^*(\Pi_x) \otimes A/\mathfrak{m}$ since A -module Π_x is flat. By Lemma 13 $H^*(\Pi_x) = 0$. The second statement follows from Theorem 7. ►

9 Extension of solutions of determined systems

Definition. Let M be a left D -module with good filtration that fulfils the condition (i). The characteristic set $V = V(\text{gr} M)$ is an algebraic cone in \mathbb{C}^n . We say that M is *underdetermined* in $V = \mathbb{C}$, *determined*, if $\dim_{\mathbb{C}} V < n$ and *overdetermined*, if $\dim_{\mathbb{C}} V < n-1$.

Let U be an open set in X and $K \Subset U$. If M is not a overdetermined module, then a solution u of M in a domain $U \setminus K$ may have nonremovable singularity on K , e.g. a fundamental solution of the operator P . A necessary condition for a solution u to have an extension to U as a solution is vanishing of some momenta. Fix a smooth density ϕ with support in an open set $V \subset U$ such that $\phi = dx$ in a neighborhood of K , take an arbitrary solution v of $P^*v = 0$ defined in a neighborhood V of K such that $\text{supp } \nabla \phi \Subset V \setminus K$ and consider the integral

$$\int_{U \setminus K} u P^*(\phi v) \tag{22}$$

Note that if u has extension to U as solution of M , we can integrate in (22) by parts and get the equation $\int \phi v P(u) = 0$. We state an inverse implication:

Theorem 15 *Let M be an elliptic D -module in X and c_x be the function in X as in Theorem 7. Let $x \in X$, $0 < r \leq c_x$, $U_x = U_x(r)$, $V_x = U_x(c_x r)$ and $K \subset V_x$ be a compact set without holes. Then an arbitrary solution of $Pu = 0$ defined in $U_x \setminus K$ has a unique extension to U_x as a solution provided the integral (22) vanishes for any smooth solution v of $P^*v = 0$ in V_x .*

◀ We may assume that $\text{supp } \phi \subset V_x$ and $\phi(x) = dx$ in a neighborhood W of K . Take a smooth function e in \mathbb{R}^n supported in W that is equal to 1 in a neighborhood W_0 of K . The function $P(eu)$ is supported in W and vanishes in W_0 . Set $\alpha = P(eu)dx$ in $V_x \setminus K$ and $\alpha = 0$ in K . We have $\alpha \in E^*(V_x)^s$ and for any solution w of the equation $P'w = 0$ in V_x

$$\alpha(w) = \int_{V_x \setminus K} P(eu)w dx = \int eu P^*(w) dx = \int u P^*(\phi w) dx$$

since the distribution $P^*(\phi w)$ is supported in $V_x \setminus W$. By the assumption the right-hand side vanishes for any w . By Theorem 7 **D** there exists a distribution $\beta \in E^*(U_x)^r$ such that $P\beta = \alpha$, that is $Pu' = 0$ in U_x where $u' \doteq eu - \beta$. The functions u and u' coincide in $U_x \setminus \text{supp } e \cup \text{supp } \beta$ and are analytic in $U_x \setminus K$, hence $u' = u$ in K since K has no holes. ▶

10 Overdetermined systems

Let P be a matrix differential operator as in (1) with analytic coefficients defined in an open set $X \subset \mathbb{R}^n$. We assign to this matrix a sheaf of differential modules in X . For this we globalize the construction of Section 7: let \mathcal{O} be the sheaf of germs of analytic functions in X and \mathcal{D} be the sheaf algebra of germs of differential operators with coefficients in \mathcal{O} . The stalk of \mathcal{D} at a point $x \in X$ is the algebra D_x as in Section 2. Let \mathcal{M} be a filtered left \mathcal{D} -module defined in open set $X \subset \mathbb{R}^n$ that can be included in a strict exact sequence of filtered left \mathcal{D} -modules

$$\mathcal{D}^\sigma \xrightarrow{P} \mathcal{D}^\rho \xrightarrow{\pi} \mathcal{M} \rightarrow 0 \quad (23)$$

where $\mathcal{D}^\sigma, \mathcal{D}^\rho$ denote some filtrations in free left \mathcal{D} -modules defined as in Section 3. Here P acts as a morphism of left \mathcal{D} -modules: $a \mapsto aP$ and the filtration in \mathcal{M} is the image of the filtration in \mathcal{D}^ρ : $\mathcal{M}_k = \pi(\mathcal{D}_k^\rho)$, $k \in \mathbb{Z}$. Note that for any function \mathcal{O} -sheaf \mathcal{F} in X the space $\text{Hom}_{\mathcal{O}}(\mathcal{M}, \mathcal{F})$ is isomorphic to the space of solutions of the equation (1) in the space $\Gamma(X, \mathcal{F})$.

Proposition 16 *For any compact set $K \subset X$, the sequence (23) can be completed to a strict exact complex of \mathcal{D} -sheaves*

$$\dots \rightarrow \mathcal{D}^\tau \xrightarrow{Q} \mathcal{D}^\sigma \xrightarrow{P} \mathcal{D}^\rho \xrightarrow{\pi} \mathcal{M} \rightarrow 0 \quad (24)$$

defined in a neighborhood of K where \dots, \mathcal{D}^τ are filtered free \mathcal{D} -sheaves of the same type.

◀ Let D_X be the sheaf in X whose stalk are the algebras D and D_X^ω be the graded D_X -sheaf where ω is an arbitrary shift vector. Consider the sequence of graded D_X -modules

$$D_X^\sigma \xrightarrow{P} D_X^\rho \xrightarrow{\pi} \text{gr}\mathcal{M} \rightarrow 0$$

generated by (23). It is exact since of Proposition 9. For $k = 0, 1, 2, \dots$ we consider \mathcal{O} -sheaf $(\text{Ker } P)_k : (D_X^\sigma)_k \rightarrow (D_X^\rho)_k$. It is a coherent analytic sheaf in the real domain X . Let L be a compact set in X such that $K \subseteq L$. By the classical theory of coherent sheaves the sheaf $(\text{Ker } P)_k$ is generated in each point $x \in L$ by a finite set S_k of its sections. The total set $S \doteq \cup_k S_k$ generates D_x -sheaf $\text{Ker } P_x : D_x^\sigma \rightarrow D_x^\rho$ at each point $x \in L$. On the other hand, for any point x there is a finite subset $q_x \subset S$ that generates the stalk $(\text{Ker } P)_x$ since the algebra D_x is Noetherian. Obviously the set q_x generates the sheaf $\text{Ker } P$ also in a neighborhood of x . Therefore there is a finite set $F \subset L$ such that the union $q_L = \cup \{q_x, x \in F\}$ generates the D -sheaf $\text{Ker } P$ at each point $x \in L$. Let D_X^t be a free D -sheaf with generators e_1, \dots, e_t . Consider a D_X -morphism $Q : D_X^t \rightarrow D_X^\sigma$ such that $q_j = Q(e_j)$, $j = 1, \dots, t$ are all elements of the set q_L . Define a filtration D_X^τ in D_X^t by means of a shift vector $\tau = (\tau_1, \dots, \tau_t)$ where $\tau_j = \deg q_j$, $j = 1, \dots, t$. The morphism $Q : D^\tau \rightarrow D^\rho$ agrees with the filtrations and $\text{Im } Q = \text{Ker } P$. Next we consider the restriction Q_L of Q_X to L and repeat these arguments for the D -sheaf $\text{Ker } Q_L$ and so on. We obtain in this way an exact sequence of D_Y -sheaves

$$\dots \xrightarrow{R_x} D_Y^\tau \xrightarrow{Q_x} D_Y^\sigma \xrightarrow{P} D_Y^\rho \xrightarrow{\pi} \text{gr}\mathcal{M}_Y \rightarrow 0$$

defined in a neighborhood Y of K . Then we construct a strict exact sequence (24) by means of arguments of Propositions 10. ▶

Note that for any point $x \in X$ the stalks of (24) form a resolution of D -module \mathcal{M}_x like (14).

Theorem 17 *Suppose that a filtered left D -module \mathcal{M} in X can be included in a strict exact sequence of the form (23) such that the D_x -module $\text{gr}\mathcal{M}_x$ fulfils the condition (ii), is elliptic and overdetermined at any point $x \in X$. Let Y be a relatively compact subset of X and S be a closed C^1 -submanifold of Y of dimension $d = n - \max_X \dim_{\mathbb{C}} V(\text{gr}\mathcal{M}_x) - 2$. There exists an open neighborhood V of S such that any solution u of (1) in $Y \setminus \bar{V}$ has a unique extension in X as a solution of the same system.*

Remark. If X is connected the function $x \mapsto \dim_{\mathbb{C}} V(\text{gr}\mathcal{M}_x \otimes \mathbb{C})$ is constant in X . This follows from (ii).

Example 1. Let $d = 0$, then the statement tells that for any point $x \in X$ there exists a compact neighborhood $K_x \subset X$ such that arbitrary solution defined in $X \setminus K$ is uniquely extended to a solution in X .

◀ Proof of Theorem. Introduce an Euclidean structure in \mathbb{R}^n .

Lemma 18 *There exist positive constants $b \leq c < 1$ that depends only on K such that for an arbitrary subspace Z in \mathbb{R}^n of dimension d and arbitrary balls*

$Y(r)$ and $Z(s)$ of radius r in $Y = Z^\perp$, respectively in Z of radii $r, s \leq 1$, $cr \leq s$ such that $Y(r) \times Z(s) \subset K$ an arbitrary solution u of \mathcal{M} defined in the set $Y(r) \setminus Y(br) \times Z(s)$ has a unique extension to $Y(r) \times Z(s)$ as a solution of M .

Here and later we denote by $Y(r')$ the ball with the same center as $Y(r)$; notation $Z(s')$ has a similar meaning. To prove the Theorem we take for an arbitrary point $x_0 \in S$ the tangent subspace Z to S at x_0 and set $Y = Z^\perp$. In the case $d = 0$ we take $Z = 0, Y = \mathbb{R}^n$. Choose a positive number r such that $Y(r) \setminus Y(br) \times Z(cr) \subset X \setminus S$. This choice is possible since S is contained in $o(r)$ -neighborhood of Z . By Lemma 18 any solution u can be extended to the set $Y(r) \times Z(cr)$. This set contains a neighborhood of x_0 . We take for V the union of these neighborhoods for all $x_0 \in S$ and complete the proof of Theorem.

►

◄ Proof of Lemma 18. Choose some positive numbers $r_0, s_0 \leq 1$ such that $Y(r_0) \times Z(s_0) \Subset X$; we may assume that $r_0 = 1, s_0 = c$ by coordinate rescaling. Set $b = c^{m+1}$, $c = \inf_U c_x/4$ where c_x is the function as in Lemma 14. Choose a coordinate system (y, z) in $Y \times Z$ such that the centers of $Y(r)$ and $Z(r)$ are in the origins.

Take a smooth function e in Y with support in $Y(2b)$ such that $e = 1$ in $Y(b + \varepsilon)$ and set $v(x) = P_0(e(y)u(x))$, the function v_0 is extended by zero to $Y(b) \times Z(c)$. Take a convex polytope $\Pi \subset Z(c) \setminus Z(c/2)$; let $F_\alpha, \alpha \in N$ be its faces. Let N_k be the subset of N of faces F_α of dimension $k = 0, 1, \dots, \dim Z$; the face Π is the only one of dimension $d \doteq \dim Z$. The notation α_k always will mean that $\alpha_k \in N_k$. We suppose that each face F_{α_k} of dimension $k < d$ is a simplex and the inequality holds

$$2b \leq \text{diam } F_{\alpha_1} \leq 3b \quad (25)$$

for each 1-face. We call k -flag any sequence $A = (\alpha_k, \alpha_{k+1}, \dots, \alpha_{d-1})$ such that $F_{\alpha_k} \subset F_{\alpha_{k+1}} \subset \dots \subset F_{\alpha_{d-1}}$. For a set $G \subset Z$ and a positive ε we denote by $(G)_\varepsilon$ the open ε -neighborhood of G .

Take a smooth function f_0 in Z with compact support in $(\Pi)_b$ such that $f_0 = 1$ in Π . For an arbitrary $k < d$ and $\alpha_k \in N_k$ we choose a smooth function f_{α_k} that fulfils

- I. $\text{supp } f_{\alpha_k} \subset (F_{\alpha_k})_b$ and
- II. $\sum_{\alpha_k} f_{\alpha_k} = 1$ in $(\cup_{\alpha_k} F_{\alpha_k})_b$.

Take an arbitrary k -flag $A = (\alpha_k, \alpha_{k+1}, \dots)$ and define the function

$$v_A = P_{d-k+1}(f_{\alpha_k} \dots P_2(f_{\alpha_{d-1}} P_1(f_0 v_0)) \dots) \quad (26)$$

where P_1, \dots, P_{d+1} are differential operators as in (19) (strokes are omitted).

Lemma 19 III. *The function v_A is supported by $(F_{\alpha_k})_b$.*

IV. *For any $k+1$ -flag B we have*

$$\sum_{\alpha_k} v_{\alpha_k, B} = 0$$

where the sum is taken for all k -flags that contain B .

◀ Statement **III** follows from **I** and equation **IV** follows from **II**:

$$\sum_{\alpha_k} v_{\alpha_k, B} = P_{d-k+1} \sum_{\alpha_k} f_{\alpha_k} v_B = P_{d-k+1} v_B = 0 \quad \blacktriangleright$$

For any 1-flag $A = (\alpha_1, \dots)$ we have $v_A = v_{\alpha_0, A} + v_{\beta_0, A}$ where $\alpha_0, \beta_0 \in N_0$ are the vertices of the face F_{α_1} hence (α_0, A) and (β_0, A) are 0-flags. By **III** we have $\text{supp } v_{\alpha_0, A} \subseteq (F_{\alpha_0})_b$ and similarly for $v_{\beta_0, A}$. the left inequality (25) implies that the supports of the distributions $v_{\alpha_0, A}$ and $v_{\beta_0, A}$ are disjoint. The formula (26) yields $P_{d+1} v_{\alpha_0, A} = P_{d+1} v_{\beta_0, A} = 0$ hence by Lemma 14 there exist solutions to the equations

$$v_{\alpha_0, A} = P_d w_{\alpha_0, A}, \quad v_{\beta_0, A} = P_d w_{\beta_0, A} \quad (27)$$

with compact supports $\text{supp } w_{\alpha_0, A} \subseteq (F_{\alpha_0})_{b/2c}$, $\text{supp } w_{\beta_0, A} \subseteq (F_{\beta_0})_{b/2c}$. Set $w_A = w_{\alpha_0, A} + w_{\beta_0, A}$ and have $P_d w_A = v_A$. By (25) for any α_0 , $\text{supp } w_{\alpha_0, A} \subseteq (F_{\alpha_1})_{b/2c} \subset (F)_{3b+b/2c} \subset (F)_{b/c}$ since $3b + b/2c \leq b/c$. By **IV** we have

$$\sum_{\alpha_0, \alpha_1} v_{\alpha_0, \alpha_1, B} = \sum_{\alpha_1} v_{\alpha_1, B} = 0$$

where the sum is taken over all flags that contain the 2-flag $B = (\alpha_2, \alpha_3, \dots)$. Therefore we can assume that also

$$\sum_{\alpha_0, \alpha_1} w_{\alpha_0, \alpha_1, B} = \sum_{\alpha_1} w_{\alpha_1, B} = 0 \quad (28)$$

Define $v'_A = v_A - \sum w_{\beta, A}$ for any 1-flag A . By (25) we have $\text{supp } v'_A \subseteq (F_{\alpha_0})_{b/c}$ for any 1-flag A and an arbitrary vertex F_{α_0} of the face F_{α_1} . Due to (27) we have $P_d v'_A = 0$, hence by Lemma 14 there exists a solution w_A to $P_{d-1} w_A = v'_A$ with compact support in $(F_{\alpha_0})_{b/2c^2}$. Set for any 2-flag B

$$v'_B = v_B - \sum_{\alpha_1} w_{\alpha_1, B}$$

where the sum is taken over all 1-flags that contains the flag B . We have $\text{supp } v'_B \subseteq (F_{\alpha_0})_{b/c^2}$. By (28) and (4) we have

$$P_{d-1} v'_B = P_{d-1} \sum_{\alpha_1} (f_{\alpha_1} v_B - w_{\alpha_1, B}) = \sum_{B \subset A} (v_A - P_{d-1} w_A) = \sum_{\alpha_1} w_{\alpha_1, B} = 0$$

By Lemma 14 we can solve the equation $P_{d-2} w_B = v'_B$ for a function w'_B with compact support in $(F_{\alpha_0})_{b/2c^3}$ under the condition

$$\sum_{\alpha_2} w_{\alpha_2, C} = 0$$

for any 3-flag C . Set

$$v'_C = v_C - \sum w_{\alpha_2, C}$$

and have $P_{d-2}v'_C = 0$ for any 3-flag C . Continuing arguing in this way $d-1$ time, we get the function

$$v'_0 = v_0 - \sum_{\alpha_{d-1}} w_{\alpha_{d-1}}$$

where $\text{supp } w_{\alpha_{d-1}} \in (F_{\alpha_0})_{b/c^{d-1}}$ and $P_1 v'_0 = 0$. We have $\text{supp } v'_0 \in Y(2c) \times Z(c)$ and $v'_0 = v_0$ in $Y \times Z(c)$. We apply again Lemma 14 and find a solution to the equation $P_0 w_0 = v'_0$ with compact support in $Y(1/2) \times Z(1/2)$. We have $P_0 w_0 = f_0 P_0(eu)$ in $Y \times Z(c)$. Because of $f_0 = 1$ in $Y \times \Pi$ we have $P_0(eu - w_0) = 0$, hence $P_0((1-e)u + w_0) = 0$ in $Y \times \Pi$. The function $U \doteq (1-e)u + w_0$ fulfils the equation $P_0 U = 0$ and coincides with u in $Y(1) \setminus Y(1/2) \times Z(c-\varepsilon)$. By uniqueness of analytic continuation we have $U = u$ in $Y(1) \setminus Y(c) \times Z(c-\varepsilon)$ for arbitrary $\varepsilon > 0$. ►

Example 2. The statement of Theorem 17 does not hold in general for $\dim S = \dim_X V(\text{gr } M) + 1$. Let $\mathbb{R}^n = Y \oplus Z$, where Z is spanned by the coordinates x_1, \dots, x_d . Consider the D -module $M = D/(p_0, p_1, \dots, p_d)$ where

$$p_0 = p_0(\partial_{x_{d+1}}, \dots, \partial_{x_n}), \quad p_i = \partial_{x_i}, i = 1, \dots, d$$

where p_0 is an elliptic operator with constant coefficients in Y . It is an elliptic module and $V(\text{gr } \mathcal{M}_x) = \{(x, \xi); \xi_1 = \dots = \xi_d = p_0(\xi_{d+1}, \dots, \xi_n) = 0\}$ for any $x \in \mathbb{R}^n$. Dimension of this characteristic manifold is equal to $n-d-1$ however there is no compulsory extension for solutions of M from $\mathbb{R}^n \setminus Z$ on \mathbb{R}^n since any fundamental solution E of p_0 considered as a function in \mathbb{R}^n has singularity in Z .

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